STUDY ON ELASTIC STABILITY OF BARS SYSTEM STRUCTURE CONSIDERING ACTUAL STIFFNESS OF CONNECTIONS

NGHIÊN CỨU ỔN ĐỊNH ĐÀN HỒI CỦA KẾT CẤU HỆ THANH CÓ XÉT ĐẾN ĐỘ CỨNG THỰC TẾ CỦA CÁC LIÊN KẾT

MSc. Tran Van Bon, Assoc. Prof. Dr. Doan Van Duan1

ABSTRACT: Usually when calculating of bars system structure stability, it is assumed that the connections at the ends of the bars, the intersections between the bars are absolutely rigid or ideal pinned. In fact, the stiffness of the connections varies between the two states mentioned above. The actual stiffness of the connections has a great influence on the strength, stiffness and stability of the bars. Therefore, in this paper, the author proposes how to apply the forced displacement method to determine the Euler critical forces for straight bars subjected to longitudinal bending, considering to the actual stiffness of the connections.

TÓM TẮT: Thông thường khi tính toán ổn định của kết cấu hệ thanh, người ta giả thiết liên kết tại hai đầu thanh, các nút giao giữa các thanh là tuyệt đối cứng hoặc là khớp lý tưởng. Thực tế các liên kết có độ cứng nằm trong khoảng giữa hai trạng thái nói trên. Độ cứng thực tế của liên kết có ảnh hưởng nhiều đến độ bền, độ cứng cũng như ổn định của thanh. Vì vậy, trong bài báo này tác giả đề xuất cách áp dụng phương pháp chuyển vị cưỡng bức để xác định các lực tới hạn Euler đối với các thanh thẳng chịu uốn dọc có xét đến độ cứng thực tế của các liên kết.

TỪ KHÓA: Lực tới hạn Euler, Độ cứng thực tế của các liên kết, ổn định thanh ...

KEYWORDS: *Euler critical force, Actual stiffness of the connections, bar stability ...*

Received: *February 25, 2026; Revised: March 24, 2026; Accepted: April 26, 2026.*

1. PROBLEM STATEMENT

Frame structures are commonly used in civil and industrial construction, as well as transportation and hydraulic engineering projects. When frames have large lengths and small cross-sections, the slenderness ratio is significant; therefore, the elastic stability problem of frames needs to be studied both theoretically and experimentally.

The forced displacement method [2, 4] allows us to immediately obtain the oscillation frequencies of the bars. It differs from traditional methods, for example, it differs from the Rayleigh method [6] which only gives us the fundamental oscillation frequency, or from the commonly used method today which is to convert the determinant of the oscillation equation into diagonal form to take the product of that term to give us a polynomial equation that determines the eigenvalues. This method often uses algorithms such as the Choleski transform, the Jacobi transform, or other very complex transforms [8]. The differential equation of the stability problem is also a homogeneous differential equation, meaning there is no right-hand side.

Typically, when calculating the stability of a beam system, it is assumed that the connections at the ends of the beams and the intersections between the beams and columns are either absolutely rigid or ideal hinges. In reality, the stiffness of the connections varies between these two states. The actual stiffness of the connections significantly affects the strength, stiffness, and stability of the beam. Therefore, in this paper, the author proposes applying the forced displacement method to determine the critical Euler forces for frames systems subjected to longitudinal bending, considering the actual stiffness of the connections.

2. BAR ELEMENT MODEL WITH CONSIDERING ACTUAL STIFFNESS OF CONNECTIONS

Assume that: The connecting spring element has zero length and an anti-rotation stiffness k; Loads are applied at the frame nodes; The effects of axial and shear forces on the deformation of the connection and the bar before the critical state are ignored; The effect of axial force on the bending deformation of the bar when the system becomes unstable is not ignored.

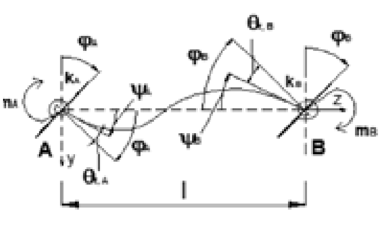


Figure 1. Beam element model considering the actual stiffness of the connections.

Consider beam AB (Figure 1) connected to the column by rotating springs of zero length and rotational stiffness at ends A and B are k_A, k_B respectively; φ_A, φ_B are the absolute rotation angles of the column ends; φ_A, φ_B are the absolute rotation angles of the beam ends; $\theta_{r,A}$, $\theta_{r,B}$ are the relative rotation angles between the beam ends and the column ends; M_A, M_B are the beam end moments at nodes A and B, we have the relationship:

$$\left.\begin{aligned} \theta_{r,A} &= \varphi_A - \psi_A = y'_{A,column} - y'_{A,beam} = \frac{1}{k_A} M_A \\ \theta_{r,A} &= \varphi_B - \psi_B = y'_{B,column} - y'_{B,beam} = -\frac{1}{k_B} M_B \end{aligned}\right\} \quad (1)$$

[1]Faculty of Engineering - Vietnam Maritime University
Email: duandv.ct@vimaru.edu.vn

These are the constraints at both ends of the bar and the frame joint.

3. STABILITY PROBLEM OF A STRAIGHT BAR SUBJECTED TO LONGITUDINAL BENDING

Consider a straight beam subjected to an axial compressive load P, with a constant bending stiffness EJ and arbitrary connections. Separate a component of length dx from the system; when this component is subjected to compression, it undergoes bending deformation as shown in Figure 2.

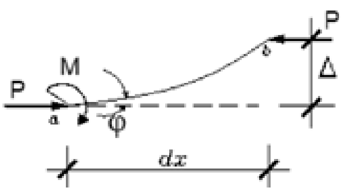


Figure 2. Beam element

At end (a) there is an internal force M causing bending deformation:

$$\chi = -EJ\frac{d^2y}{dx^2} \qquad (2)$$

Assume that at end (b) there is a displacement Δ,

At end (a), there is an external force Mp=PΔ causing a rotation angle:

$$\varphi = \frac{dy}{dx} \qquad (3)$$

Since φ is small, so ,

$$tg\varphi = \varphi \to \Delta = \varphi dx = \frac{dy}{dx}dx$$

we have:

$$M_P = P\Delta = P\frac{dy}{dx}dx$$

According to the Gaussian principle of extremum, the forced quantity of the problem can be written as follows:

$$Z = \int_0^1 M\left(-\frac{d^2y}{dx^2}\right)dx - \int_0^1 P\frac{dy}{dx}\left(\frac{dy}{dx}\right)dx \to \min \qquad (4)$$

From the extreme conditions of the beam, we have:

$$\delta Z = -\delta\left[\int_0^1 M\left(\frac{d^2y}{dx^2}\right) + \int_0^1 P\left(\frac{d^2y}{dx^2}\right)\right]dx \to \min \qquad (5)$$

Or:

$$-\frac{d^2}{dx^2}[M-P] = 0 \Leftrightarrow -\frac{d^2M}{dx^2} + P\frac{d^2y}{dx^2} = 0$$

Substitute $\chi = -EJ\frac{d^2y}{dx^2}$ into the above equation, we have: $EJ\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0$ (6)

Equation (6) is the stability equation of a straight bar subjected to longitudinal bending. Thus, from the Gaussian extreme principle method, we also obtain the stability equations of a bar subjected to longitudinal bending, similar to other principles. Equation (6) is the equilibrium differential equation of a bar subjected to longitudinal bending by a force P applied at the end of the bar. It is a homogeneous linear differential equation (without a right-hand side) whose traditional method of solving it along with boundary conditions has been presented in [3]. Below is the method of forced displacement to solve equation (6).

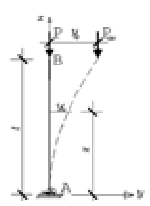
4. FORCED DISPLACEMENT MEHOD

The forced displacement method aims to transform equation (6) which is the equation of balance between internal and external forces into an equation with a right-hand side by giving a point in the bar, for example point x = x1, a displacement y0, figure 3: $g = y_{x=x1} - y_0 = 0$ (7)

Transform the problem of finding the extremum of (4) with the constraint condition (7) into an unconstrained extremum problem by constructing the extended Lagrange functional F as follows:

$$F = Z + \lambda g \to \min \qquad (8)$$

Figure 3. Fixed end bar - free end



$$F = \int_0^1 M\left(-\frac{d^2y}{dx^2}\right)dx - \int_0^1 P\frac{dy}{dx}\left(\frac{dy}{dx}\right)dx + \lambda\left(y_{x=x1} - y_0\right) \to m \qquad (9)$$

where: λ - the Lagrange factor and also the unknown of the problem. From the condition $\delta F = \int_0^1 (M - M_P)\delta[\chi]dx + \delta(\lambda g) = 0$ (10)

We obtain the following equation:

$$EJ\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = \begin{cases} -\lambda \text{ khi } x=x1 \\ 0 \text{ khi } x\neq x1 \end{cases} \qquad (11)$$

along with equation (7). Equation (11) is the equation with the right side. For it to become the longitudinal bending equation (6) of the bar, then: λ(P)=0 (12)

Mathematically, equation (11) is a polynomial equation that determines the eigenvalues of system (6) because its solution is also the solution of (6). Mechanically, λ has the unit of force. It is the holding force so that the bar has a displacement y0 at point x = x1. The holding force must be zero, hence equation (12). The eigenvalue of (6) depends on the parameter P, hence λ is also a function of P. Therefore, solving equation (12) in terms of P will receive the critical forces of the longitudinally bent bar.

5. DETERMINING THE CRITICAL FORCE OF A BAR AND BAR SYSTEM UNDER COMPRESSION, CONSIDERIG THE STIFFNESS OF THE CONNECTIONS

Example 1. Fixed-end, free-end bar

Given a straight bar of length l, bending stiffness EJ=Const, subjected to an axial compressive force P applied at one end of the bar. At end A, the bar is connected by an elastic rotating spring with an initial stiffness kA=∞ (fixed end), and end B is free, as shown in Figure 3. Determine the critical force for the bar in cases where the spring stiffness changes.

The steps to solve the problem are as follows:

Step 1: Write the expression for the deflection curve of the bar

In this exercise, we approximate the elastic curve of the bar in the form of a polynomial as follows:

$$y_1 = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \quad \text{(a1)}$$

where a_i (i=1÷7) are the coefficients to be determined

Let χ be the bending strain in the bar, according to (2) we have: $\chi = -EJ\frac{d^2y}{dx^2}$

Thus, the bending moment M_x in the bar will be equal to:

The force P will cause a bending moment in the bar of:

$$M_P = P(y_1 - y_0)$$

Step 2: Write the expression for the forced quantity Z according to (3)

$$Z = \int_0^1 (M - M_P)[\chi]dx \to \min \qquad \text{(b1)}$$

With the constraints, the number of constraints depends on each specific problem, in this case we have the following constraints:

Displacement at the end of the bar is y0; Bending moment at the end of the bar is zero; The rotation angle at the fixed end is written according to (1), this is a constraint considering the actual connection.

where $r = \frac{EJ}{lk_A}$; k_A is the rotational stiffness of the spring;

$$\left.\begin{aligned} g_1 &= y_1\big|_{x1=l} - y_0 = 0 \\ g_2 &= -EJ\frac{d^2y_1}{dx^2}\bigg|_{x1=l} = 0 \\ g_3 &= \frac{dy_1}{dx}\bigg|_{x=0} + r\frac{d^2y_1}{dx^2}\bigg|_{x=0} = 0 \end{aligned}\right\} \text{(c1)}$$

Step 3: Write the extended functional expression F according to (8)

We transform the extremum problem (b1) with three constraints above into an extremum problem without constraints by introducing the Lagrange factor into the extended functional as follows:

$$F = Z + \sum_{k=1}^{3} g_k \lambda_k \rightarrow \min \quad \text{Or:}$$

$$F = \int_0^1 (M = M_P)[\chi]dx + g_1\lambda_1 + g_2\lambda_2 + g_3\lambda_3 \rightarrow \min \quad \text{(d1)}$$

where λ_1 is the Lagrange factor and also the unknown of the problem, which is the force holding the system in the skewed state. The problem has 10 unknowns: the coefficients of the polynomial (a1), the ai (i=1÷7) and the Lagrange factors λ_k(k=1÷3).

Step 4: Establish the system of linear algebraic equations and solve the system.

Gaussian's extremum principle considers bending strains to be independent of the applied moment, so the extremum condition of the extended functional F is:

$$\left.\begin{aligned} h_i &= \int_0^1 (M = M_P)\frac{\partial}{\partial a_i}[\chi]dx + \frac{\partial}{\partial a_i}\sum_{k=1}^{3}(g_k\lambda_k) = 0 \\ f_i &= \frac{\partial}{\partial \lambda_k}\sum_{k=1}^{3}(g_k\lambda_k) = 0; \quad a_i = 1 \div 7; \lambda_k(k = 1 \div 3) \end{aligned}\right\} \quad \text{(e1)}$$

Thus, from the extreme conditions of the extended functional F, we will obtain 10 linear algebraic equations to determine the unknowns. The above problem can be solved using Matlab's Symbolic software. After solving the equations, we see that the parameters ai (i=1÷7) and λ_k(k=1÷3) are both functions of force P. Here, we only introduce the Lagrange factor λ_1; in the case r=0, corresponding to stiffness $k_A=\infty$ (ideal fixation at the base of column A), we have:

Step 5: Derive the equation with the unknown variable λ being the force holding the system in the skewed state and solve the equation to find the critical forces.

In this case, it is λ1, we have: $\lambda 1=7/45x(46xP^5xl^{10}-1035xP^5xrl^9-22455xEJxP^4xl8+305280xEJxP^4rl^7+3295080xEJ2xP^3xl^6+-27038880xEJ^2xP^3rl^5-169884000xEJ^3xP^2xl1^4+795484800xEJ^3P^2rl^3+2630232000xEJ^4xPxl^2-5504241600xEJ^4xPrl^2-5504241600xEJ^5)y0=0$ (f1)

We see that λ1 is a 5th-degree polynomial of P. Solving (f1) we will get 5 solutions, the values of the solutions change depending on the stiffness k of the spring, in the equation r appears, r and k are related according to expression (1). The first three critical forces of the rod are as follows: P_{1th}= 2.4674xEJ/l^2; P_{2th}=22.2070xEJ/l^2; $P3_{th}$=61.8805xEJ/l^2.

The first three critical forces are completely accurate compared to the results obtained when solved using traditional methods.

Table 1. By varying the spring stiffness k from 0 to ∞, we obtain the following results:

$r=\dfrac{EJ}{lk_A}$	According to the forced displacement method		According to traditional methods
	P_{1th}	P_{2th}	P_{1th}
0	2.467	22.207	2.467
1.0	7.835	37.549	
∞	9.878	39.648	9.869

From the tabulated results, we see that when r=0, corresponding to the rotational stiffness of the spring k=∞, the connection at the column base is an ideal fixed support as commonly encountered. The critical force obtained by the forced displacement method completely matches the results obtained when solving using traditional methods. As r gradually increases,

corresponding to the decreasing stiffness k, we see that the critical force increases rapidly until r=10, at which point the rate of increase slows down. Gradually, until r=∞ (k=0), at which point the column base connection becomes an ideal hinge, the critical force in this case is the critical force of the two ends of the ideal hinge connection.

Example 2. Single-story, single-span rigid frame

Give the rigid frame as shown in Figure 4, the lengths of the bars are $l_1=l_2=l^3=l$, the bending stiffness EJ=Const, subjected to axial compression force P applied at the left column head. At the column bases A, D and nodes B, C there are elastic rotating spring connections with corresponding initial stiffness $k_A=\infty$, $k_D=\infty$ (two fixed column bases), $k_B=\infty$, $k_C=\infty$ (two rigid nodes), as shown in Figure 4.

The requirement is to determine the critical force for the frame in cases where the spring stiffness varies, and from there find the critical force for frames with frame joints and ground connections that are ideal connections commonly encountered in calculations.

The convention for coordinates and displacement direction of the bars in the frame is the same as in problem 1.

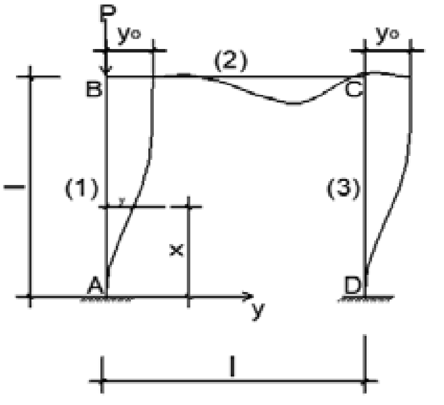


Figure 4. Third-order statically indeterminate frame

Approximate the elastic curve shapes y1, y2, y3 corresponding to the three bar segments 1, 2, and 3 in polynomial form similar to problem 1. Then, construct the extreme problem with the following 7 constraints:

$$\left.\begin{aligned} & g_1 = \left(\frac{dy_1}{dx}\right)\Big|_{x=0} - r_1\left(\frac{d^2y_1}{dx^2}\right)\Big|_{x=0} = 0; \quad g_2 = \left(\frac{dy_2}{dx}\right)\Big|_{x=0} - \left(\frac{dy_1}{dx}\right)\Big|_{x=l} - r_2\left(\frac{d^2y_2}{dx^2}\right)\Big|_{x=0} = 0 \\ & g_3 = y_2\big|_{x=l} = 0; \quad g_4 = \left(\frac{dy_2}{dx}\right)\Big|_{x=l} - \left(\frac{dy_3}{dx}\right)\Big|_{x=l} + r_3\left(\frac{d^2y_2}{dx^2}\right)\Big|_{x=l} = 0 \\ & g_5 = y_3\big|_{x=l} - y_1\big|_{x=l} = 0; \quad g_6 = \left(\frac{dy_3}{dx}\right)\Big|_{x=0} - r_4\left(\frac{d^2y_3}{dx^2}\right)\Big|_{x=0} = 0; \quad g_7 = y_1\big|_{x=l} - y_0 = 0 \end{aligned}\right\}$$

By constructing the extended functional and solving the extreme problem, we have:

$\lambda_7=-.60000x10^{12}y_0EJ(.42188\ x10^{22}P^8l^{16}-.18886\ x10^{26}P^7l^{14}EJ+.22566\ x10^{29}P^6l^{12}EJ^2-.34022\ x10^{39}xl^2EJ^7P+.25509\ x10^{34}P^4l^8EJ^4+.15240\ x10^{38}l^4EJ^6P^2-.28732\ x10^{36}P^3l^6EJ^5-.11052\ x10^{32}P^5l^{10}EJ^3+.25034\ x10^{40}EJ^8)=0$

We see that in this case, λ7 is a polynomial of degree 8 of P. Solving λ7=0) in terms of P, we will get 8 solutions. These are the critical forces Pth that we need to find. When $k_A=k_D=k_B=k_C=\infty$, corresponding to $r_1=r_2=r_3=r_4=0$, we get the critical force of the rigid frame in Figure 4, which matches the results obtained from existing methods. Here, we only give the first three solutions, respectively:

P_{1th}= 14.587 ; P_{2th}= 27.78 ; P_{3th}= 69.71 ;

- When $k_A=k_D=k_C=\infty$, $k_B=0$, the bars have bending stiffness EJ=Const as shown in Figure 5, resulting in a frame with one hinge at B and two fixed column bases at A and D.

P_{1th}= 7.77 ; P_{2th}= 23.041 ; P_{3th}= 61.990 ;

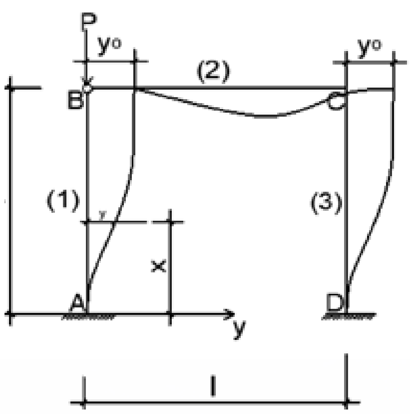


Figure 5. Second-order statically indeterminate frame

6. CONCLUSION

The author has successfully used the forced displacement method to construct and solve the elastic stability problem of a straight beam system subjected to longitudinal bending, taking into account the actual stiffness of the connections. This demonstrates the superiority of the forced displacement method for eigenvalue and eigenvector problems compared to other methods, immediately obtaining the characteristic polynomial determining the critical force of the beam system simply by assigning a certain distance y0 to any point on the beam undergoing forced displacement. This is an advantage of the method and also a new scientific contribution. The results obtained without considering the actual stiffness of the connections completely match the results obtained using existing methods.

The critical forces obtained when considering the actual stiffness of the connections show the consistency of the physical properties of the connections; increasing the rotational stiffness of the spring converges to the fixed-joint case, and conversely, decreasing the stiffness k gradually approaches the hinged-joint case. This allows the designer, using this problem, to find results for many other problems by changing the rotational stiffness of the spring. For example, in the problem in example 2, we can simultaneously obtain results for the first-order statically indeterminate frame (when stiffness $k_B=k_C=0$), the second-order statically indeterminate frame (when stiffness $k_A=k_D=k_C=\infty$, kB=0), and the third-order statically indeterminate frame (when stiffness $k_A=k_B=k_C=k_D=\infty$)...

7. RECOMMENDATIONS

It is possible to study the calculations of continuous beam systems and truss systems, taking into account the actual stiffness of the connections, for static, dynamic, and stability problems. Using the full theory of beams, we can study the internal forces and displacements, vibrations, and stability of bending structural systems, considering the actual stiffness of the connections.

This research is funded by Vietnam Maritime University under grant number: DT25-26.96

REFERENCES

[1] H. H. Cuong, "Gaussian extreme principle method," J. Sci. Technol. (Vietnam), vol. IV, pp. 112–114, 2005.

[2] D. V. Duan, "Forced displacement method for solving eigenvalues and eigenvectors," Constr. J. (Vietnam), no. 11, pp. 82–84, 2014.

[3] D. V. Duan, Study on Elastic Stability of Bar System Structures Considering Lateral Shear Strain. Hanoi, Vietnam: Construction Publishing House, 2016, 156 pp.

[4] V. T. Thuy, "Study of internal force and displacement of flexural bar systems considering the influence of shear strain," Ph.D. dissertation, Dept. Civil Eng., Hanoi Architectural University, Hanoi, Vietnam, 2010.

[5] P. V. Trung, "New method for calculating wire systems and suspended roofs," Ph.D. dissertation, Dept. Civil Eng., Hanoi Architectural University, Hanoi, Vietnam, 2006.

[6] C. Lanczos, The Variational Principles of Mechanics. Toronto, ON, Canada: University of Toronto Press, 1949.

[7] F. P. Beer, E. R. Johnston Jr., and J. T. DeWolf, Mechanics of Materials, 4th ed. New York, NY, USA: McGraw-Hill, 2006, 787 pp.